

FREE-SURFACE WAVES AND FAR WAKES GENERATED BY A FLOATING BODY IN A VISCOUS FLUID*

Lu Dong-qiang

Department of Mechanical Engineering, The University of Hong Kong, Hong Kong, China, e-mail: dqlu@mail.shu.edu.cn

Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China

(Received May 14, 2002)

ABSTRACT: The free-surface waves and the flow field due to a body moving on the surface of an incompressible viscous fluid of infinite depth were studied analytically. The floating body was modeled as a normal point pressure on the free surface. Based on the Oseen approximation for governing equations and the linearity assumption for boundary conditions, the exact solutions in integral form for the free-surface elevation, the velocities and the pressure were given. By employing Lighthill's two-stage scheme, the asymptotic representations in far field for large Reynolds numbers were derived explicitly. The effect of viscosity on the wave profiles was expressed by an exponential decay factor, which removes the singular behavior predicted by the potential theory.

KEY WORDS: free-surface waves, velocity distribution, floating body, viscosity, asymptotic representation, Lighthill's two-stage scheme.

1. INTRODUCTION

The wave motion due to a moving disturbance has been of long-standing interest. But the corresponding wake structure did not receive much attention until the synthetic aperture radar collected a number of images of V-like wake patterns generated by ships (Milgram 1988; Brown et al. 1989). The wake structures are thought to be classified into three categories: the Kelvin wave wakes, the turbulent wakes and the internal wave wakes (Griffin 1988; Lyden et al. 1988; Gu and Phillips 1994). The Kelvin wakes are the far-field wave patterns generated by the flow around the moving body (Noblesse and Chen 1995; Reed and Milgram 2002), and the Kelvin waves are found to be the

sum of two families of waves, that is, the transverse and diverging waves. These two waves are confined within a V-shaped region behind the body (Crapper 1964). However, the generation mechanism of these V-like wakes has not been fully explored.

Based on the well-known potential theory, Tao et al. (2001) analytically studied the far-field distribution of horizontal velocities on free-surface waves generated by a floating point pressure. To obtain a unique solution of physical reality, the steady-state inviscid problem is solved with a radiation condition that a local disturbance must propagate outward to infinity (Mei 1994). Another alternative mathematical device is to introduce a "fictitious viscosity" in the manner suggested by Rayleigh (Mei 1994). As the "fictitious viscosity" tends to zero, the solution of physical interest will be obtained. However, the inviscid potential theories predict infinite diverging wave amplitude (Chung and Lim 1991) and infinite horizontal velocity near the moving path of a floating point pressure (Tao et al. 2001), which is really a singular behavior from the physical point of view.

It is believed that the viscosity plays an important role in the free-surface flow since all fluids are naturally viscous. Taking an Oseen-type approximation, Cumberbatch (1965) studied the effect of viscosity on ship waves while Brard (1970) studied the effect of vorticity on ship waves. Bloor (1970) considered the effect of viscosity on long waves in shallow water. Lugt (1987) studied the local flow

* This research was jointly sponsored by the Hong Kong Research Grants Council (Grant No: HKU 7076/02E) and by Shanghai Key Subject Program.

properties at a viscous free surface by means of series expansions of the Navier-Stokes equations. Miles (1990, 1991) investigated the linear wave motion in a viscous fluid of variable depth.

Based on the assumption that the flow generated by the floating body is a laminar one, the free-surface waves, the velocities and the pressure are analytically studied in this paper. The floating body is modeled as a normal point pressure on the free surface. The general mathematical formulation is provided in Section 2. Exact solutions in the Fourier integral form are given in Section 3. Asymptotic representations are derived with the aid of Lighthill's two-stage scheme in Section 4. Physical interpretations for the characteristics of the free-surface waves and the velocity distribution are discussed in Section 5. Finally, conclusions are reached in Section 6.

2. GENERAL MATHEMATICAL FORMULATION

The free-surface flow due to a body moving with a constant velocity on the free surface of an incompressible viscous fluid of infinite depth will be considered. Cartesian coordinates are taken fixed on the body, as shown in Fig. 1. The x axis is along the straight path of the moving body while the z axis points vertically upward. Thus, the fluid is moving at uniform velocity Ue_x , where U is a constant and e_x the unit vector in the x direction. It is assumed that all motions arrive in a steady state.

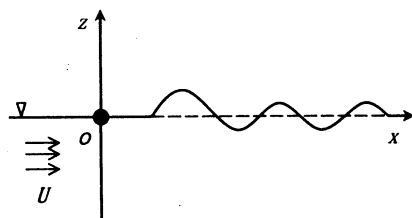


Fig. 1 Definition coordinate system

We choose ρU^2 as a reference pressure, and U^2/g as a characteristic length. Where ρ is the density of the fluid and g the gravitational acceleration. For simplicity, the velocity is nondimensionalized by U , the pressure by ρU^2 , and the distance by U^2/g . The governing equations are the dimensionless continuity equation and the steady Oseen equations,

$$\nabla \cdot \mathbf{u} = 0 \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial x} = -\nabla(p + z) + \epsilon \nabla^2 \mathbf{u} \quad (2)$$

where $\mathbf{u} = (u, v, w)$ is the disturbed velocity vector, p the pressure, and

$$\epsilon = \mu g / \rho U^3 \quad (3)$$

where μ is the dynamic viscosity of the fluid. The dimensionless parameter ϵ can be regarded as the reciprocal of the Reynolds number with respect to the characteristic length.

For small-amplitude waves, linearized boundary conditions are imposed at the undisturbed free surface ($z = 0$),

$$\frac{\partial \eta}{\partial x} = w \quad (4)$$

$$\epsilon \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0 \quad (5)$$

$$\epsilon \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0 \quad (6)$$

$$-p + 2\epsilon \frac{\partial w}{\partial z} = -F_z \delta(x) \delta(y) \quad (7)$$

where η is the dimensionless elevation of the free surface. Eq. (4) is the kinematic boundary condition at the free surface. Eq. (5) represents the vanishing of shearing stress in the x direction while Eq. (6) in the y direction. Eq. (7) denotes that the normal stress on the free surface balances the applied singular force with the dimensionless magnitude F_z . Eqs. (5) to (7) are referred as the dynamic boundary conditions.

According to the theorem of splitting for the Oseen equations (Lagerstrom 1996), a solution of Eqs. (1) and (2) may be decomposed into a longitudinal component $(\mathbf{u}_L, p)$ and a transversal component $(\mathbf{u}_T, 0)$

$$\mathbf{u} = \mathbf{u}_L + \mathbf{u}_T \quad (8)$$

where $(\mathbf{u}_L, p)$ and $(\mathbf{u}_T, 0)$ satisfy the following equations respectively:

$$\nabla \cdot \mathbf{u}_L = 0 \quad (9)$$

$$\nabla \times \mathbf{u}_L = 0 \quad (10)$$

$$\frac{\partial \mathbf{u}_L}{\partial x} = -\nabla(p+z) \quad (11)$$

$$\nabla \cdot \mathbf{u}_T = 0 \quad (12)$$

$$\frac{\partial \mathbf{u}_T}{\partial x} = \epsilon \nabla^2 \mathbf{u}_T \quad (13)$$

Eq. (10) implies that there exists a potential Φ , where such that

$$\mathbf{u}_L = \nabla \Phi \quad (14)$$

It follows from Eq. (9) that

$$\nabla^2 \Phi = 0 \quad (15)$$

From Eq. (11), we obtain

$$p = -\frac{\partial \Phi}{\partial x} - z \quad (16)$$

We express the free surface boundary conditions in term of Φ and $\mathbf{u}_T$ as

$$\frac{\partial \eta}{\partial x} - \frac{\partial \Phi}{\partial z} - w_T = 0 \quad (17)$$

$$2 \frac{\partial^2 \Phi}{\partial x \partial z} + \frac{\partial u_T}{\partial z} + \frac{\partial w_T}{\partial x} = 0 \quad (18)$$

$$2 \frac{\partial^2 \Phi}{\partial y \partial z} + \frac{\partial v_T}{\partial z} + \frac{\partial w_T}{\partial y} = 0 \quad (19)$$

$$\frac{\partial \Phi}{\partial x} + \eta + 2\epsilon \left(\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial w_T}{\partial z} \right) = -F_z \delta(x) \delta(y) \quad (20)$$

where u_T , v_T , w_T are the components of $\mathbf{u}_T$.

3. EXACT SOLUTIONS

Taking the Fourier transform with respective to x and y over Eqs. (15) and (13) yields

$$\frac{\partial^2 \bar{\Phi}}{\partial z^2} + (\alpha^2 + \beta^2) \bar{\Phi} = 0 \quad (21)$$

$$i\alpha \bar{\mathbf{u}}_T = -\epsilon(\alpha^2 + \beta^2) \bar{\mathbf{u}}_T + \epsilon \frac{\partial^2 \bar{\mathbf{u}}_T}{\partial z^2} \quad (22)$$

where $\bar{\Phi}(\alpha, \beta, z)$ and $\bar{\mathbf{u}}_T(\alpha, \beta, z)$ denote the Fourier transforms of $\Phi(x, y, z)$ and $\mathbf{u}_T(x, y, z)$, respectively. The solutions of Eqs. (21) and (22) are

$$\bar{\Phi} = \tilde{\Phi}(\alpha, \beta) \exp(Az) \quad (23)$$

$$\bar{\mathbf{u}}_T = \tilde{\mathbf{u}}_T(\alpha, \beta) \exp(Bz) \quad (24)$$

$$A = \sqrt{\alpha^2 + \beta^2} \quad (25)$$

$$B = \sqrt{i\alpha/\epsilon + \alpha^2 + \beta^2} \quad (26)$$

$\tilde{\Phi}$ is a scalar function and $\tilde{\mathbf{u}}_T = (\tilde{u}_T, \tilde{v}_T, \tilde{w}_T)$ a vector, which are independent of z . Eqs. (23) and (24) are chosen to ensure the vanishing of the solutions as $z \rightarrow -\infty$. Substitution of Eqs. (23) and (24) into the Fourier transforms of Eq. (12) and Eqs. (17) to (20) yields a set of simultaneous algebraic equations, which can readily be solved. Thus, the wave profile, the potential function, the velocities for the transversal component and the pressure can be given by the double Fourier integral

$$\mathbf{V}_1 = \frac{F_z}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{D} \mathbf{V}_2 \exp(i\alpha x + i\beta y) d\alpha d\beta \quad (27)$$

where

$$\mathbf{V}_1 = \{\eta, \Phi, u_T, v_T, w_T, p\} \quad (28)$$

$$\mathbf{V}_2 = \{-A, -i\alpha - 2A^2\epsilon, 2i\alpha A B \epsilon, 2i\beta A B \epsilon, 2A^3\epsilon, -\alpha^2 + 2i\alpha A^2\epsilon\} \quad (29)$$

$$D = A - \alpha^2 + 4i\alpha A^2\epsilon + 4A^3(A - B)\epsilon^2 \quad (30)$$

4. ASYMPTOTIC REPRESENTATIONS

What interests us in this problem is to find the far-field behavior of the surface elevation, the velocities and the pressure distributions, which are exactly represented in Eq. (27). To obtain an explicit physical interpretation for Eq. (27), we shall employ Lighthill's two-stage scheme (Lighthill 1960), which in essence involves calculating the α integration by the residue theorem and the β inte-

gration by the method of steepest descent. The reciprocal of the Reynolds number will be used as the perturbation parameter throughout the asymptotic analysis.

To obtain the asymptotic expansion of Eq. (27), the analytic nature of D in Eq. (30) shall be investigated. For small ϵ , Eq. (30) has two simple zeros with respect to α ,

$$\alpha_n = (-1)^{n+1} a_0 + i\epsilon a_1 + O(\epsilon^{3/2}), \quad (n = 1, 2) \quad (31)$$

where

$$a_0(\beta) = [1 + (1 + 4\beta^2)^{1/2}]^{1/2} / \sqrt{2} \quad (32)$$

$$a_1(\beta) = 4a_0^5(2a_0^2 - 1)^{-1} \quad (33)$$

For an inviscid flow, the zeros were found at $\alpha_n = (-1)^{n+1} a_0$ by Crapper (1964). The first term in Eq. (31) represents an inviscid contribution while the second one a viscous contribution. In accordance with Lighthill's theory, the contribution from poles of the integrand dominates the asymptotic behavior of double integrals in Eq. (27).

We introduce the polar coordinates (R, θ) on the horizontal (x, y) plane by

$$x = R \cos \theta, \quad y = R \sin \theta \quad (34)$$

For the first stage of Lighthill's scheme, Cauchy's residue theorem is employed. The leading term for the wave profile at large R is given by

$$\eta = \frac{F_z}{2\pi} \sum_{n=1}^2 \int_{-\infty}^{\infty} \frac{i(-1)^n a_0^3}{1 - 2a_0^2} \exp[Rf_n(\beta)] d\beta + O(R^{-1}) \quad (35)$$

where

$$f_n(\beta) = i\alpha_n \cos \theta + i\beta \sin \theta \quad (36)$$

The asymptotic location of the zeros in the upper half-plane, as shown in Eq. (31), ensures that there is no wave contribution from $x < 0$.

For the second stage of Lighthill's scheme, the method of steepest descent for large R should be employed. The saddle points in β plane are determined from

$$\frac{\partial f_n(\beta)}{\partial \beta} = 0 \quad (37)$$

The leading term of the asymptotic solution of Eq. (37) is given by

$$\beta = (-1)^n q_m Q_m^{1/2} \tan \theta, \quad (n, m = 1, 2) \quad (38)$$

where

$$q_m = 2[1 + (-1)^{m+1} \sqrt{1 - 8 \tan^2 \theta}]^{-1} \quad (39)$$

$$Q_m = (q_m + 1)/2 \quad (40)$$

Upon a steepest-descent analysis and other mathematical manipulation, the wave profile can be formally expressed as

$$\eta(R, \theta) = -F_z d_m^R \sum_{m=1}^2 d_m^V Q_m^{5/4} \sin M_m + O(R^{-1}) \quad (41)$$

where

$$d^R = \left(\frac{2}{\pi R \cos \theta} \right)^{1/2} (1 - 8 \tan^2 \theta)^{-1/4} \quad (42)$$

$$d_m^V = \exp(-4\epsilon q_m^{-1} Q_m^3 R \cos \theta) \quad (43)$$

$$M_m = Q_m^{1/2} R \cos \theta - q_m Q_m^{1/2} R \sin \theta \tan \theta + (-1)^{m+1} \frac{\pi}{4} \quad (44)$$

The approach to obtain the asymptotic solutions for the velocity field and the pressure in far wakes is similar to that for free-surface waves, and the procedures will not be reproduced herein. Accordingly, the formal expressions for the velocities and the pressure can be given by

$$u_L = F_z d^R \sum_{m=1}^2 d_m^V Q_m^{5/4} (\sin M_m - 2\epsilon Q_m^{3/2} \cos M_m) + O(R^{-1}) \quad (45)$$

$$v_L = -F_z d^R \sum_{m=1}^2 d_m^V Q_m^{5/4} q_m (\sin M_m - 2\epsilon Q_m^{3/2} \cos M_m) \tan \theta + O(R^{-1}) \quad (46)$$

$$w_L = -F_z d^R \sum_{m=1}^2 d_m^V Q_m^{7/4} (\cos M_m + 2\epsilon Q_m^{3/2} \sin M_m) + O(R^{-1}) \quad (47)$$

$$u_T = 2\epsilon^{1/2} F_z d^R \sum_{m=1}^2 d_m^V Q_m^2 \cos(M_m + \pi/4) + O(\epsilon R^{-1}) \quad (48)$$

$$v_T = -2\epsilon^{1/2} F_z d^R \sum_{m=1}^2 d_m^V Q_m^2 q_m \cos(M_m + \pi/4) \cdot \tan \theta + O(\epsilon R^{-1}) \quad (49)$$

$$w_T = 2\epsilon F_z d^R \sum_{m=1}^2 d_m^V Q_m^{13/4} \sin M_m + O(\epsilon R^{-1}) \quad (50)$$

$$p = -F_z d^R \sum_{m=1}^2 d_m^V Q_m^{5/4} (\sin M_m - 2\epsilon Q_m^{3/2} \cdot \cos M_m) + O(\epsilon R^{-1}) \quad (51)$$

5. PHYSICAL INTERPRETATIONS

From Eqs. (45) to (50), an immediate conclusion can be drawn that in the limit of vanishing viscosity, the present results for a viscous fluid will simply reduce to those for the corresponding inviscid fluid. It is believed that the terms of order ϵ in the longitudinal and transversal components are created due to the presence of viscosity. These terms will totally vanish as the parameter ϵ tends to zero. Thus, the free-surface elevation and the longitudinal component will reduce to those obtained by Tao et al. (2001) for an inviscid fluid. Furthermore, it can be seen that there exists a phase shift of $\pi/2$ between the leading and secondary terms in the longitudinal component.

The values of $Q_1(\theta)$ and $Q_2(\theta)$ are shown in Fig. 2. The expressions involving $Q_1(\theta)$ correspond to the so-called transverse wave system while the expressions involving $Q_2(\theta)$ the so-called diverging wave system. The effect of viscosity on the wave profile and the velocity distribution is expressed by the exponential decay factors in Eq. (43). The viscous decay factors on the transverse and diverging waves are shown in Figs. 3 and 4, respectively. It can be seen that the diverging waves are more severely damped than the transverse waves. Thus, the singularity-induced wave system consists prima-

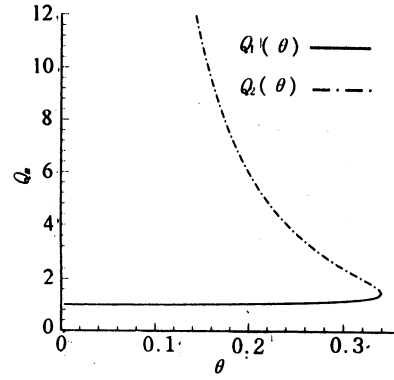


Fig. 2 The values of $Q_1(\theta)$ and $Q_2(\theta)$

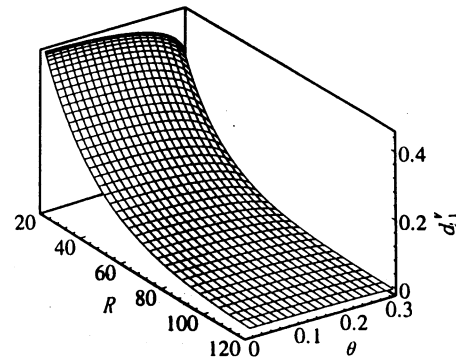


Fig. 3 The viscous damping factor for the transverse waves at $\epsilon = 0.01$

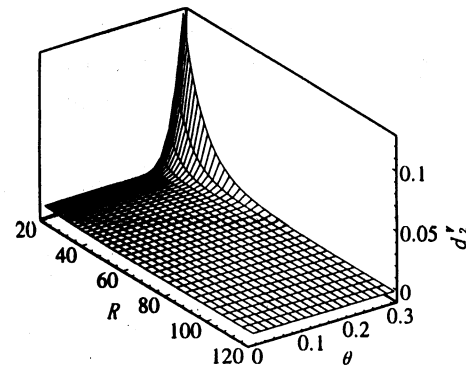


Fig. 4 The viscous damping factor for the diverging waves at $\epsilon = 0.01$

rially of the transverse waves. It should be noted that the potential theories predict infinite diverging wave amplitude along the path of the floating body (i.e. $\theta = 0$). However, the viscous theory predicts zero diverging wave amplitude at $\theta = 0$, which is reasonable from the physical point of view. Fig. 5 shows the free-surface profiles generated by the moving point pressure with $F_z = 4\pi$ and ϵ

$= 0.01$. All waves are confined within a V-shaped region

$$\Delta_7 = \{(R, \theta) \mid 0 < R < \infty, |\theta| < \tan^{-1}\sqrt{1/8}\} \quad (52)$$

which is consistent with the inviscid case.

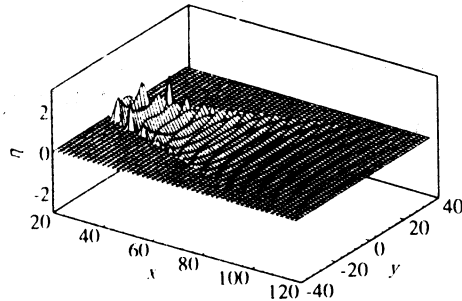


Fig. 5 The free-surface profile at $F_z = 4\pi$ and $\epsilon = 0.01$

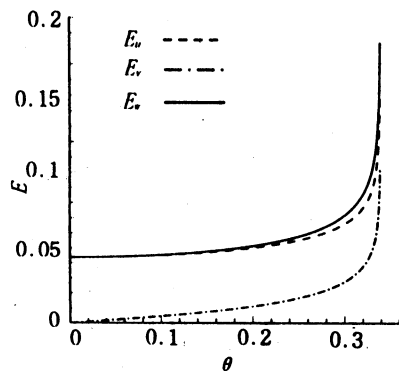


Fig. 6 The velocity envelopes at $R = 30$ and $\epsilon = 0.01$

Eqs. (45) to (51) show that the analytical forms for the velocity and the pressure are very similar to those for the wave profiles. The longitudinal component of the velocity is found to be the dominant one. To study the main behavior of the velocity distribution, the velocity envelopes for the longitudinal components are defined as follows

$$E_u = d^R d_m^V Q_m^{5/4} \quad (53)$$

$$E_v = d^R d_m^V q_m Q_m^{5/4} \tan \theta \quad (54)$$

$$E_w = d^R d_m^V Q_m^{7/4} \quad (55)$$

The envelope profiles for the velocities at $R = 30$ and ϵ are shown in Fig. 6. It can be seen that velocity v_L is smaller than u_L and w_L . The velocity in the

y direction is zero along the line $\theta = 0$ since the velocity field is symmetric about $\theta = 0$. The velocity amplitudes increase as the angle θ increases, which is consistent with the characteristic of the wave amplitude.

6. CONCLUSIONS

Analytical solutions have been obtained for the free-surface waves and the flow field generated by a body moving on the surface of an incompressible viscous fluid of infinite depth. With the aid of Lighthill's two-stage scheme, the asymptotic representations for the free-surface elevation, the velocity distribution and the pressure distribution in the far field wave have been explicitly derived. It has been found that the viscosity plays a significant role both in the mathematical formulation and in the physical interpretation for the free-surface elevation and the velocity distribution. The well-known results for the inviscid case can be covered by the present work in the limit of vanishing viscosity.

ACKNOWLEDGMENTS

The author is grateful to Professor Allen T. Chwang for his valuable suggestion and discussion.

REFERENCES

- [1] BLOOR M. I. G. Effect of viscosity of long waves in shallow water[J]. *Phys. Fluids*, 1970, 13(6): 1435-1439.
- [2] BRARD R. Viscosity, wake and ship waves[J]. *J. Ship Res.*, 1970, 14(4): 207-240.
- [3] BROWN E. D., BUCHSBAUM S. B. et al. Observations of a nonlinear solitary wave packet in the Kelvin wake of a ship[J]. *J. Fluid Mech.*, 1989, 204: 263-293.
- [4] CHUNG Y. K. and LIM J. S. A review of the Kelvin ship wave pattern [J]. *J. Ship Res.*, 1991, 35(3): 191-197.
- [5] CRAPPER G. D. Surface waves generated by a traveling pressure point[J]. *Proc. Roy. Soc. Lond. A*, 1964, 282: 547-558.
- [6] CUMBERBATCH E. Effects of viscosity on ship waves [J]. *J. Fluid Mech.*, 1965, 23: 471-479.
- [7] GRIFFIN O. M. Ship wave modification by a surface current field[J]. *J. Ship Res.*, 1988, 32(3): 186-193.
- [8] GU D. F. and PHILLIPS O. M. On narrow V-like ship wakes[J]. *J. Fluid Mech.*, 1994, 275: 301-321.
- [9] LAGERSTROM P. A. *Laminar Flow Theory* [M]. Princeton: Princeton University Press, 1996.

- [10] Lighthill M. J. Studies on magneto-hydrodynamic waves and other anisotropic wave motions[J]. **Phil. Trans. Roy. Soc. Lond. A**, 1960, 252: 397-430.
- [11] LUGT H. J. Local flow properties at a viscous free surface[J]. **Phys. Fluids**, 1987, 30(12): 3647-3652.
- [12] LYDEN J. D., HAMMOND R. R. et al. Synthetic aperture radar imaging of surface ship waves[J]. **J. Geophys. Res.**, 1988, 93(C10): 12293-12303.
- [13] MEI C. C. **The Applied Dynamics of Ocean Surface Waves** (Second printing with corrections) [M]. Singapore: World Scientific Publishing, 1994.
- [14] MILES J. Wave motion in viscous fluid of variable depth [J]. **J. Fluid Mech.**, 1990, 212: 365-372.
- [15] MILES J. Wave motion in viscous fluid of variable depth Part 2. moving contact line[J]. **J. Fluid Mech.**, 1991, 223: 47-55.
- [16] MILGRAM J. H. Theory of radar backscatter from short waves generated by ships, with application to radar (SAR) imagery[J]. **J. Ship Res.**, 1988, 32(1): 54-69.
- [17] NOBLESSE F. and CHEN X. B. Decomposition of free-surface effects into wave and near-field components[J]. **Ship Technol. Res.**, 1995, 42: 167-185.
- [18] REED A. M. and MILGRAM J. H. Ship wakes and their radar images[J]. **Annu. Rev. Fluid Mech.**, 2002, 34: 469-502.
- [19] TAO Mingde, LIU Minjia and REN Jiuchen. A study of velocity field in ship waves[J]. **Journal of Hydrodynamics, Ser. B**, 2001, 13(4): 42-47.

CIMS Papers' World

Here scientific papers on Computer Integrated Manufacturing System(CIMS) and other fields!for you FREELY!

1.<http://www.cimspaper.com>
2.<http://xcchen.yeah.net>

3.<http://cimspaper.icpcn.com>
4.<http://cimspaper.51.net>